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Approximation of Continuous and Quasi-Continuous Functions by Monotone Functions

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Let Q denote the Banach space (sup norm) of quasi-continuous functions defined on the interval $[0, 1]$. Let C denote the subspace comprised of continuous functions. Met M denote the closed convex cone in Q comprised of nondecreasing functions. For $f \in Q$ and $1 < p < \infty$, let f_p denote the best L_p-approximation to f by elements of M. It is shown that f_p converges uniformly as $p \to \infty$ to a best L_∞-approximation to f by elements of M. If $f \in C$, then each $f_p \in C$; so $f_\infty \in C$.

We begin with some introductory remarks and notation. By a function, unless we specify otherwise, we mean a real-valued function defined on the interval $[0, 1]$.

A function f is in Q if and only if (a) $f(x^+) = \lim_{y \to x^+} f(y)$ exists, $0 \leqslant x < 1$, and (b) $f(x^-) = \lim_{y \to x^-} f(y)$ exists, $0 < x \leqslant 1$.

Let P denote the set of partitions $\pi = \{t_i\}_{i=0}^n$ of $[0, 1]$ (i.e., $0 = t_0 < t_1 < \cdots < t_n = 1$), let I_E denote the indicator function of a subset E of $[0, 1]$ (i.e., $I_E(x) = 1$ if $x \in E$ and $I_E(x) = 0$ otherwise), and let S denote the dense linear subspace of Q comprised of simple step functions $f = \sum_{i=0}^n a_i I_{[t_i]} + \sum_{i=1}^n b_i I_{(t_{i-1}, t_i)}$.

Consider the elements f of Q as bounded Lebesgue measurable functions, and let $[f] = \{g\colon g \text{ is measurable}, f = g \text{ a.e.}\}$ be the corresponding element of L_∞. A function f in Q is zero a.e. $\Leftrightarrow f(x^+) = f(x^-) = 0$, $0 < x < 1$. Thus, if we let Q^* denote the space of functions f in Q such that $f(0) = f(0^+)$ and $f(x) = f(x^-)$, $0 < x \leqslant 1$, then we have a linear isometry between Q^* and the embedding of Q in L_∞. This isometry permits us to restrict our attention to Q^*, so we let $M^* = M \cap Q^*$ and $S^* = S \cap Q^*$. Thus, $f \in S^*$ if there exists $\pi \in P$ such that $f = a_1 I_{[t_0, t_1]} + \sum_{i>1} a_i I_{(t_{i-1}, t_i]}$.

For a bounded function f and $\pi \in P$, $\bar{f}_\pi \in S^*$ is defined by

$$\begin{aligned} \bar{f}_\pi(x) &= \sup\{f(y)\colon y \in [t_0, t_1]\}, && x \in [t_0, t_1], \\ &= \sup\{f(y)\colon y \in (t_{i-1}, t_i]\}, && x \in (t_{i-1}, t_i], \quad i > 1; \end{aligned}$$

$\underline{f}_\pi$ is defined by replacing sup with inf.

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A bouned function f is in Q^* if and only if for each $\varepsilon > 0$ there exists $\pi \in P$ such that $0 \leqslant \bar{f}_\pi - \underline{f}_\pi < \varepsilon$. Thus, $\lim_\pi \bar{f}_\pi = \lim_\pi \underline{f}_\pi = f$ (uniformly), $f \in Q^*$, where $\lim_\pi$ denotes the Moore–Smith or directed set limit. The fact that S^* is dense in Q^* permits us to use a result of Ubhaya for functions defined on a finite partially ordered set.

Because L_p is a uniformly convex Banach space, $1 < p < \infty$, for each $f \in Q^*$ there is a unique nearest point $f_p \in M^*$. We show that f_p converges uniformly as $p \to \infty$ to a best L_∞-approximation f_∞ to f by elements of M^*.

After establishing convergence of $\{f_p\}_{p>1}$, we conclude with two examples. The first example shows how f_∞ compares with the set of all best L_∞-approximations to f, and the second example points out that f_∞ and f_p (for large p) may increase while f_2 is a constant function. The latter example suggests that the presence of a trend in a data sequence may depend on how one defines trend.

To establish convergence of $\{f_p\}_{p>1}$, we recall the following theorem of Ubhaya [2]:

Let $X = \{x_1, x_2, ..., x_n\}$ be a finite partially ordered set and let $f = \{f_i\}_{i=1}^n = \{f(x_i)\}_{i=1}^n$ be a real-valued function defined on X. For each p, $1 < p < \infty$, define a weighted p-norm of f by

$$\|f\|_{\omega,p} = \left[\sum_{i=1}^{n} \omega_{p,i} |f_i|^p\right]^{1/p}, \tag{1}$$

where $\omega_p = \{\omega_{p,i}\}_{i=1}^n > 0$ is a given weight function defined on X. Similarly, if $\omega = \{\omega_i\}_{i=1}^n > 0$ is a weight function, define the weighted uniform norm $\| \|_\infty$ of f by

$$\|f\|_\infty = \max_{1 \leqslant i \leqslant n} \omega_i |f_i|. \tag{2}$$

Definition. A subset $L \subseteq X$ is a lower set if $x_i \in L$ and $x_j \in X$, $x_j \leqslant x_i$ implies that $x_j \in L$. Similarly a subset $U \subseteq X$ is an upper set if $x_i \in U$ and $x_j \in X$, $x_j \geqslant x_i$ implies that $x_j \in U$.

Definition. Let $\mathscr{M}$ denote the class of monotone increasing functions on X, i.e., the function $h = \{h_i\}_{i=1}^n \in \mathscr{M}$ if $h(x_i) = h_i \leqslant h_j = h(x_j)$ whenever x_i, $x_j \in X$ and $x_i \leqslant x_j$.

Fact 1 (Ubhaya). Let $f = \{f_i\}_{i=1}^n$ be fixed. For each p, $1 < p < \infty$, let $g_p = \{g_{p,i}\}_{i=1}^n$ be the function defined on X by

$$\begin{aligned} g_{p,i} &= \max_{\{U: i \in U\}} \min_{\{L: i \in L\}} U_p(L \cap U) \\ &= \min_{\{L: i \in L\}} \min_{\{U: i \in U\}} U_p(L \cap U), \end{aligned} \tag{3}$$

where L and U are lower and upper sets, respectively, and $U_p(L \cap U)$ is the unique real number minimizing $\sum_{j \in L \cap U} \omega_{p,j} |f_j - u|^p$. Then g is the unique monotone increasing function satisfying

$$\|f - g\|_{\omega,p} = \inf\{\|f - h\|_{\omega,p} : h \in \mathscr{M}\}$$

or

$$\left[\sum_{i=1}^{n} \omega_{p,i} |f_i - g_{p,i}|^p \right]^{1/p} = \inf \left\{ \left[\sum_{i=1}^{n} \omega_{p,i} |f_i - h_i|^p \right]^{1/p} : \{h_i\}_{i=1}^{n} \in \mathscr{M} \right\}. \quad (4)$$

THEOREM 1 (Ubhaya). *Let X and f be as defined above. For each p, $1 < p < \infty$, let $\omega_p = \{\omega_{p,i}\}_{i=1}^{n} > 0$ be a weight function and assume that there exists a weight function $\omega = \{\omega_i\}_{i=1}^{n} > 0$* such that

$$0 < \lim_{p \to \infty} \inf(\omega_{p,i}/\omega_i^p) \leqslant \lim_{p \to \infty} \sup(\omega_{p,i}/\omega_i^p) < \infty \quad (5)$$

for all i. Then the monotone increasing functions g_p, $1 < p < \infty$, defined by (3) *and satisfying* (4) *converge as $p \to \infty$ to a monotone increasing function $g_\infty = \{g_{\infty,i}\}_{i=1}^{n}$ which satisfies*

$$\|f - g\|_\infty = \inf\{\|f - h\|_\infty : h \in \mathscr{M}\}$$

or

$$\max_{1 \leqslant i \leqslant n} \omega_i |f_i - g_{\infty,i}| = \inf\{\max_{1 \leqslant i \leqslant n} \omega_i |f_i - h_i| : \{h_i\}_{i=1}^{n} \in \mathscr{M}\}. \quad (6)$$

Moreover, for every $i \leqslant n$

$$\begin{aligned} g_{\infty,i} = \lim_{p \to \infty} g_{p,i} &= \max_{\{U : i \in U\}} \min_{\{L : i \in L\}} U_\infty(L \cap U) \\ &= \min_{\{L : i \in L\}} \max_{\{U : i \in U\}} U_\infty(L \cap U), \end{aligned} \quad (7)$$

where $U_\infty(L \cap U)$ is the unique real number minimizing $\max_{j \in L \cap U} \omega_j |f_j - u|$ for all real u.

Remark 1. Notice that if there exist real numbers δ, ρ such that $0 < \delta \leqslant \omega_{p,i} < \rho$ for all p and all i, then clearly (5) holds if and only if $\omega_i = 1$ for all i; or else if $\omega_i < 1$, then $\omega_i^p \to 0$ as $p \to \infty$, and if $\omega_i > 1$, then $\omega_i \to \infty$ as $p \to \infty$. In either case, (5) can not be satisfied. In our application of Theorem 1, $\omega_{p,i} = t_i - t_{i-1}$ and $\omega_i = 1$, $i \leqslant n$.

LEMMA 1. *If $f \in S_\pi^*$, then $f_p \in S_\pi^*$ for all p, $1 < p < \infty$.*

Proof. Suppose that f_p is not a constant a.e. on some subinterval $(t_{j-1}, t_j]$. Then let

$$l = \operatorname{essinf}\{f_p(t) : t_{j-1} < t \leqslant t_j\}$$

and

$$u = \operatorname{essup}\{f_p(t) : t_{j-1} < t \leqslant t_j\}.$$

Clearly $l < u$. Choose $\xi \in [l, u]$ such that

$$|f_j - \xi| = \inf\{|f_j - r| : r \in [l, u]\}.$$

Then the monotone increasing function f_p^* defined by

$$\begin{aligned} f_p^*(t) &= \xi, \qquad && t_{j-1} < t \leqslant t_j, \\ &= f_p(t), && \text{otherwise}, \end{aligned}$$

is a better best L_p-approximation to f since

$$\begin{aligned} \|f - f_p^*\| &= \left[\sum_{\substack{i=1 \\ i \neq j}}^{n} \int_{t_{i-1}}^{t_i} |f_i - f_p(t)|^p \, dt + \int_{t_{j-1}}^{t_j} |f_j - \xi|^p \, dt \right]^{1/p} \\ &< \left[\sum_{\substack{i=1 \\ i \neq j}}^{n} \int_{t_{i-1}}^{t_i} |f_i - f_p(t)|^p \, dt + \int_{t_{j-1}}^{t_j} |f_j - f_p(t)|^p \, dt \right]^{1/p} \end{aligned}$$

or

$$\|f - f_p^*\|_p < \|f - f_p\|_p.$$

This contradiction shows that f_p must have a constant value everywhere on $(t_{j-1}, t_j]$ and hence $f_p \in S_\pi^*$.

THEOREM 2. *Let $f \in S_\pi^*$ be given by*

$$f = f_1 I_{[0, t_1]} + \sum_{i=2}^{n} f_i I_{(t_{i-1}, t_i]}. \tag{8}$$

For every p, $1 < p < \infty$, let $\omega_p = \{\omega_{p,i}\}_{i=1}^n$ be defined by

$$\omega_{p,i} = t_i - t_{i-1} \tag{9}$$

for all i. Let $g_p = \{g_{p,i}\}_{i=1}^n$ be as defined by (3). Then f_p is given by

$$f_p = g_{p,1} I_{[0, t_1]} + \sum_{i=2}^{n} g_{p,i} I_{(t_{i-1}, t_i]}. \tag{10}$$

Proof. By the last lemma we have $f_p \in S_\pi^*$. For every i, let

$$x_i = (t_i + t_{i-1})/2, \qquad i = 1, 2,..., n,$$

and let $X = \{x_1,..., x_n\}$. Consider $\{f_i\}_{i=1}^n$ as a finite real-valued function defined on X and let $\{h_i\}_{i=1}^n$ ($h_i \leqslant h_j$ for all $i < j$) be a monotone increasing function on X. Then by substituting the values of $\omega_{p,i}$ in Eq. (4) we conclude that

$$\left[\sum_{i=1}^n (t_i - t_{i-1})\,|f_i - g_{p,i}|^p\right]^{1/p} \leqslant \left[\sum_{i=1}^n (t_i - t_{i-1})\,|f_i - h_i|^p\right]^{1/p}$$

or

$$\left[\sum_{i=1}^n \int_{t_{i-1}}^{t_i} |f_i - g_{p,i}|^p\right]^{1/p} \leqslant \left[\sum_{i=1}^n \int_{t_{i-1}}^{t_i} |f_i - h_i|^p\right]^{1/p},$$

which is equivalent to the conclusion that

$$\|f - f_p\|_p \leqslant \|f - h\|_p,$$

where

$$h = h_1 I_{[0,t_1]} + \sum_{i=2}^n h_i I_{(t_{i-1},t_i]}$$

is any monotone increasing function belonging to S_π^*. ∎

THEOREM 3. *Let $f \in S_\pi^*$ and let f_p be as given in Theorem 2. Then f_p converges as $p \to \infty$ to the monotone increasing function $f_\infty \in S_\pi^*$ given by*

$$f_\infty = g_{\infty,1} I_{[0,t_1]} + \sum_{i=2}^n g_{\infty,i} I_{(t_{i-1},t_i]}, \tag{11}$$

where $g_{\infty,i} = \lim_{p\to\infty} g_{p,i}$ is given by (7). Moreover, f_∞ is a best L_∞-approximation to f by monotone increasing functions.

Proof. Let X and ω_p be as defined above. To apply Theorem 1, observe that (5) holds if and only if $\omega_i = 1$ for all i (see Remark 1). In this case, the theorem implies that $g_p = \{g_{p,i}\}_{i=1}^n$ converges to $g_\infty = \{g_{\infty,i}\}_{i=1}^n$ which is given by (7). Therefore, $\lim_{p\to\infty} f_p$ exists and it is given by (11).

For the last part of the theorem, substitute for the values of ω_i in (6) to obtain

$$\max_{1\leqslant i\leqslant n} |f_i - g_{\infty,i}| \leqslant \max_{1\leqslant i\leqslant n} |f_i - h_i|, \qquad \{h_i\}_{i=1}^n \in \mathscr{M}. \tag{12}$$

Thus, f_∞ is a best L_∞-approximation to f by elements of S_π^*. Let h be a monotone increasing function defined on Ω. We show that there is a monotone increasing function $g \in S_\pi^*$ such that

$$\|f-g\|_\infty \leqslant \|f-h\|_\infty.$$

Indeed for $i = 1, 2,..., n$, let

$$g_i = \{\tfrac{1}{2}[\operatorname{essup}(k(x)) + \operatorname{essinf}(h(x))]: t_{i-1} < x \leqslant t_i\}.$$

Then clearly

$$|f_i - g_i| \leqslant \operatorname{essup} |f_i - h(x)|, \qquad t_{i-1} < x \leqslant t_i,$$

for all i. Now define g on Ω by

$$g = g_1 I_{[0,t_1]} + \sum_{i=2}^{n} g_i I_{(t_{i-1},t_i]}.$$

Then $g \in S_\pi^*$ and it follows from the last inequality together with (12) that

$$\|f-f_\infty\|_\infty \leqslant \|f-g\|_\infty \leqslant \|f-h\|_\infty.$$

This concludes the proof.

Remark 2. Let $f \in S^*$. Then there is a partition π of Ω such that $f \in S_\pi^*$. Using Lemma 1 and the conclusions of Theorems 2 and 3, we find the best L_p-approximations f_p, $1 < p < \infty$, to f by monotone increasing functions. Then we showed that the monotone increasing function $f_\infty = \lim_{p\to\infty} f_p$ is well defined.

To put this another way, denote f by f_π to indicate that $f \in S_\pi^*$. Similarly, let

$$f_{\pi,p} = (f_\pi)_p. \tag{13}$$

Then

$$f_{\pi,\infty} = (f_\pi)_\infty = \lim_{p\to\infty} f_{\pi,p} \tag{14}$$

is well defined.

Next, we generalize these results to Q^*, the space of all quasi-continuous real-valued functions which are left continuous at every point of Ω except at 0, where they are right-continuous. We start with

Remark 3. (a) Let f and g be elements of Q^*. Then it is shown in [1, p. 366, Theorem 3(ii)] that if $f \leqslant g$, then

$$f_p \leqslant g_p \tag{15}$$

for all p, $1 < p < \infty$,

(b) It is clear that for any constant c and for all $f \in Q^*$ we have

$$(f+c)_p = f_p + c \tag{16}$$

for all p, $1 < p < \infty$.

DEFINITION. Let $f \in Q^*$ and let $\pi = \{t_i\}_{i=0}^n$ be a partition of Ω. The oscillation of f over $[t_0, t_1]$ is defined by

$$\tilde{O}(f, [t_0, t_1]) = \sup\{(f(x) - f(y)) : x, y \in [t_0, t_1]\}$$

and for $i = 2, 3,..., n$; the oscillation of f over $(t_{i-1}, t_i]$ is defined by

$$\tilde{O}(f, (t_{i-1}, t_i]) = \sup\{(f(x) - f(y)) : x, y \in (t_{i-1}, t_i]\}.$$

Finally, we define the oscillation of f over π by

$$\tilde{O}(f, \pi) = \max\{\tilde{O}(f, [t_0, t_1]), \tilde{O}(f, (t_{i-1}, t_i]) : i = 2, 3,..., n\}. \tag{17}$$

LEMMA 2. *Let* $\pi' = \{t_i'\}_{i=0}^{n'}$ *be a refinement of* $\pi = \{t_i\}_{i=0}^n$ *(written* $\pi < \pi'$*). Then*

$$\tilde{O}(f, \pi') \leqslant \tilde{O}(f, \pi). \tag{18}$$

Proof. Since $t_1' \leqslant t_1$, then it is clear from the above definition that

$$\tilde{O}(f, [t_0', t_1']) \leqslant \tilde{O}(f, [t_0, t_1]) \leqslant \tilde{O}(f, \pi). \tag{19}$$

Next, let $2 \leqslant k' \leqslant n'$. Then there exists some k, $1 \leqslant k \leqslant n$, such that $(t_{k'-1}', t_{k'}'] \subseteq (t_{k-1}, t_k]$. Consequently, it follows that

$$\tilde{O}(f, (t_{k'-1}', t_{k'}']) \leqslant \tilde{O}(f, (t_{k-1}, t_k]) \leqslant \tilde{O}(f, \pi).$$

By taking the sup over all k' and combining (19) we conclude (18). ∎

Remark 4. Let $f \in Q^*$ and let $\varepsilon > 0$ be given. Then there exists a partition π such that

$$\tilde{O}(f, \pi) < \varepsilon.$$

Moreover, if $0 < \varepsilon' < \varepsilon$, then we can find a refinement π' of π such that $\tilde{O}(f, \pi') < \varepsilon'$. In other words, by further refinements of π we can make $\tilde{O}(f, \pi')$ as small as we wish. We denote this by writing

$$\lim_{\pi} \tilde{O}(f, \pi) = 0.$$

DEFINITION. Let $f \in Q^*$ and let $\pi = \{t_i\}_{i=0}^n$ be a partition of Ω. Let $\bar{f}_\pi$ and $\underline{f}_\pi$ be the step functions defined by

$$\bar{f}_\pi = \bar{a}_1 I_{[t_0, t_1]} + \sum_{i=2}^{n} \bar{a}_i I_{(t_{i-1}, t_i]} \tag{20}$$

and

$$\underline{f}_\pi = \underline{a}_1 I_{[t_0, t_1]} + \sum_{i=2}^{n} \underline{a}_i I_{(t_{i-1}, t_i]}, \tag{21}$$

where

$$\bar{a}_i = \sup\{f(x) \colon t_{i-1} < x \leqslant t_i\}; \qquad \mathrm{i} = 1, 2, \ldots, n$$

and

$$\underline{a}_i = \inf\{f(x) \colon t_{i-1} < x \leqslant t_i\}; \qquad i = 1, 2, \ldots, n.$$

By Remark 2 we define

$$\bar{f}_{\pi,p} = (\bar{f}_\pi)_p, \tag{22}$$

$$\underline{f}_{\pi,p} = (\underline{f}_\pi)_p; \tag{23}$$

and

$$\bar{f}_{\pi,\infty} = (\bar{f}_\pi)_\infty = \lim_{p \to \infty} \bar{f}_{\pi,p}, \tag{24}$$

$$\underline{f}_{\pi,\infty} = (\underline{f}_\pi)_\infty = \lim_{p \to \infty} \underline{f}_{\pi,p}. \tag{25}$$

LEMMA 3. *For all p, $1 < p < \infty$, we have*

$$0 \leqslant \bar{f}_{\pi,p} - \underline{f}_{\pi,p} \leqslant \tilde{O}(f, \pi) \tag{26}$$

and

$$0 \leqslant \bar{f}_{\pi,\infty} - \underline{f}_{\pi,\infty} \leqslant \tilde{O}(f, \pi). \tag{27}$$

Proof. Let $x \in \Omega$. Then $x \in (t_{j-1}, t_j]$ for some $j \leqslant n$. Hence

$$\begin{aligned} 0 \leqslant \bar{f}_\pi(x) - \underline{f}_\pi(x) &= \sup\{f(y): t_{j-1} < y \leqslant t_j\} - \inf\{f(y): t_{j-1} < y \leqslant t_j\} \\ &= \sup\{(f(y_1) - f(y_2)): y_1, y_2 \in (t_{j-1}, t_j]\} \\ &= \tilde{O}(f, (t_{j-1}, t_j]) \leqslant \tilde{O}(f, \pi) \end{aligned}$$

or

$$\bar{f}_\pi(x) \leqslant \underline{f}_\pi(x) + \tilde{O}(f, \pi)$$

for all $x \in \Omega$. Therefore we obtain

$$\bar{f}_\pi \leqslant \underline{f}_\pi + \tilde{O}(f, \pi).$$

By (15) and (16) we obtain

$$\bar{f}_{\pi,p} \leqslant (\underline{f}_\pi + \tilde{O}(f, \pi))_p = \underline{f}_{\pi,p} + \tilde{O}(f, \pi) \tag{28}$$

or

$$0 \leqslant f_{\pi,p} - \underline{f}_{\pi,p} \leqslant \tilde{O}(f, \pi).$$

Finally, we let $p \to \infty$ to obtain (27). ∎

LEMMA 4. *Let $f \in Q^*$ and let $\pi < \pi'$. Then*

$$\underline{f}_{\pi,p} \leqslant \underline{f}_{\pi',p} \leqslant \bar{f}_{\pi',p} \leqslant \bar{f}_{\pi,p} \leqslant \underline{f}_{\pi,p} + \tilde{O}(f, \pi) \tag{29}$$

and

$$\underline{f}_{\pi,\infty} \leqslant \underline{f}_{\pi',\infty} \leqslant \bar{f}_{\pi',\infty} \leqslant \bar{f}_{\pi,\infty} \leqslant \underline{f}_{\pi,\infty} + \tilde{O}(f, \pi). \tag{30}$$

Proof. Since $\pi < \pi'$, then it clearly follows from their definitions that

$$\underline{f}_\pi \leqslant \underline{f}_{\pi'} \leqslant \bar{f}_{\pi'} \leqslant \bar{f}_\pi.$$

Thus, it follows from (15) and (28) that

$$\underline{f}_{\pi,p} \leqslant \underline{f}_{\pi',p} \leqslant \bar{f}_{\pi',p} \leqslant \bar{f}_{\pi,p} \leqslant \underline{f}_{\pi,p} + \tilde{O}(f, \pi)$$

which is (29). Letting $p \to \infty$ we obtain (30). ∎

THEOREM 4. *Let $f \in Q^*$ with best monotone L_p-approximation f_p. Then*

$$\lim_\pi \bar{f}_{\pi,p} = \lim_\pi \underline{f}_{\pi,p} = f_p. \tag{31}$$

Proof. By (29) and (26) we conclude that for $\pi < \pi'$ we obtain

$$0 \leqslant \bar{f}_{\pi,p} - \bar{f}_{\pi',p} \leqslant \bar{f}_{\pi,p} - \underline{f}_{\pi',p}$$
$$\leqslant \bar{f}_{\pi,p} - \underline{f}_{\pi,p} \leqslant \tilde{O}(f, \pi),$$

but by Remark 4 we have

$$\lim_{\pi} \tilde{O}(f, \pi) = 0,$$

so that we obtain

$$0 \leqslant \bar{f}_{\pi,p} - \bar{f}_{\pi',p} < \varepsilon$$

for every $\varepsilon > 0$ provided that π is chosen appropriately. Therefore $\lim_{\pi} \bar{f}_{\pi,p} = \bar{f}_p$ exists. Similarly, we have

$$0 \leqslant \underline{f}_{\pi',p} - \underline{f}_{\pi,p} \leqslant \bar{f}_{\pi',p} - \underline{f}_{\pi,p}$$
$$\leqslant \bar{f}_{\pi,p} - \underline{f}_{\pi,p} \leqslant \tilde{O}(f, \pi) < \varepsilon,$$

which implies that $\lim_{\pi} \underline{f}_{\pi,p} = \underline{f}_p$ exists. Applying (26) once more we conclude that $\bar{f}_p = \underline{f}_p = f_p^*$. We need to show that $f_p^* = f_p$ so let $\varepsilon > 0$ be given. Then there is a partition π such that

$$\bar{f}_{\pi} < f + \varepsilon \qquad \text{and} \qquad f < \underline{f}_{\pi} + \varepsilon,$$

which implies upon using (15) and (16) that

$$\bar{f}_{\pi,p} < f_p + \varepsilon \qquad \text{and} \qquad f_p < \underline{f}_{\pi,p} + \varepsilon.$$

Taking the limit over π, we conclude that

$$f_p^* < f_p + \varepsilon \qquad \text{and} \qquad f_p < f_p^* + \varepsilon$$

or

$$f_p = f_p^*. \quad \blacksquare$$

THEOREM 5. *Let $f \in Q^*$ with best monotone L_p-approximation f_p. Then*

$$\lim_{\pi} \bar{f}_{\pi,\infty} = \lim_{\pi} \underline{f}_{\pi,\infty} = f_{\infty} = \lim_{p \to \infty} f_p. \tag{32}$$

Proof. From (30) and (27) we obtain for $\pi < \pi'$

$$0 \leqslant \bar{f}_{\pi,\infty} - \bar{f}_{\pi',\infty} \leqslant \bar{f}_{\pi,\infty} - \underline{f}_{\pi',\infty}$$
$$\leqslant \bar{f}_{\pi,\infty} - \underline{f}_{\pi,\infty} \leqslant \tilde{O}(f, \pi) < \varepsilon$$

for an appropriate choice of π. Hence $\lim_{\pi} \bar{f}_{\pi,\infty} = \bar{f}_{\infty}$ exists.

Similarly, we have

$$0 \leqslant \underline{f}_{\pi',\infty} - \underline{f}_{\pi,\infty} \leqslant \bar{f}_{\pi',\infty} - \underline{f}_{\pi,\infty}$$
$$\leqslant \bar{f}_{\pi,\infty} - \underline{f}_{\pi,\infty} \leqslant \tilde{O}(f,\pi) < \varepsilon$$

for an appropriate choice of π. Hence $\lim_\pi \underline{f}_{\pi,\infty} = \underline{f}_\infty$ exists. Now it follows from (27) that

$$\bar{f}_\infty = \underline{f}_\infty = f_\infty.$$

We still need to show that f_p converges uniformly to f_∞. Let $\varepsilon > 0$ be given. Then for an appropriate π we have by the last theorem that

$$|f_p - \bar{f}_{\pi,p}| < \varepsilon/3$$

for all p, $1 < p < \infty$, and also we have

$$|\bar{f}_{\pi,\infty} - f_\infty| < \varepsilon/3.$$

Since $\bar{f}_{\pi,\infty} = \lim_{p\to\infty} \bar{f}_{\pi,p}$ by definition, then there exists a real number $p_0 > 1$ such that

$$|\bar{f}_{\pi,p} - \bar{f}_{\pi,\infty}| < \varepsilon/3$$

for all $p > p_0$. Combining these last three inequalities, we obtain

$$|f_p - f_\infty| \leqslant |f_p - \bar{f}_{\pi,p}| + |\bar{f}_{\pi,p} - \bar{f}_{\pi,\infty}| + |\bar{f}_{\pi,\infty} - f_\infty|$$
$$\leqslant \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

for all $p > p_0$. This completes the proof. ▮

COROLLARY 1. *Let f and g be in Q^*. Then*

(a) *if $f \leqslant g$ on Ω, then $f_\infty \leqslant g_\infty$, and*

(b) *if c is a real constant, then $(f+c)_\infty = f_\infty + c$.*

Proof. This corollary is an immediate consequence of Remark 3 and the fact that $\lim_{p\to\infty} f_p = f_\infty$. ▮

THEOREM 6. *Suppose $f \in Q^*$ is continuous. Then f_p is continuous.*

Proof. Let x be an arbitrary but fixed point in (0, 1) and let $\varepsilon > 0$ be given. Then

$$|f_p(x) - f_p(y)| \leqslant |f_p(x) - \bar{f}_{\pi,p}(x)| + |\bar{f}_{\pi,p}(x) - \bar{f}_{\pi,p}(y)|$$
$$+ |\bar{f}_{\pi,p}(y) - f_p(y)|. \tag{33}$$

By Theorem 4, we know that

$$f_p(y) = \lim_\pi \bar{f}_{\pi,p}(y)$$

for all $y \in \Omega$. Therefore we can choose $\pi = \{t_i\}_{i=0}^n$ such that

(1) Each of the first and third terms on the right-hand side of (33) is less than $\varepsilon/3$.

(2) If $\bar{f}_\pi$ can be written as

$$\bar{f}_\pi = \bar{a}_1 I_{[t_0,t_1]} + \sum_{i=2}^n \bar{a}_i I_{(t_{i-1},t_i]}, \tag{34}$$

then we can have by uniform continuity of f over Ω that

$$|\bar{a}_i - \bar{a}_{i-1}| < \varepsilon/9 \tag{35}$$

for all $i = 2, 3,..., n$.

Thus, (33) becomes

$$|f_p(x) - f_p(y)| < \varepsilon/3 + \varepsilon/3 + |\bar{f}_{\pi,p}(x) - \bar{f}_{\pi,p}(y)| \tag{36}$$

for all $y \in \Omega$. All we need now is to show the existence of a real number $\delta > 0$ such that

$$|\bar{f}_{\pi,p}(x) - \bar{f}_{\pi,p}(y)| < \varepsilon/3 \tag{37}$$

for all $y \in (x - \delta, x + \delta)$. To show this we first observe that if $\bar{f}_\pi$ is given by (34), then $\bar{f}_{\pi,p}$ must be given by

$$\bar{f}_{\pi,p} = b_1 I_{[t_0,t_1]} + \sum_{i=2}^n b_i I_{(t_{i-1},t_i]} \tag{38}$$

for some real numbers $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n$. We now have two cases to consider.

Case 1: $t_{j-1} < x < t_j$ for some $j \leqslant n$. Then it follows that

$$|\bar{f}_{\pi,p}(x) - \bar{f}_{\pi,p}(y)| = |b_j - b_j| = 0$$

for all $y \in (t_{j-1}, t_j]$. Let $\delta = \min\{(x - t_{j-1}), (t_j - x)\}$. Then (36) becomes

$$|f_p(x) - f_p(y)| < (2\varepsilon/3) + 0 < \varepsilon$$

for all $y \in (x - \delta, x + \delta)$ which implies the continuity of f_p at x in this case.

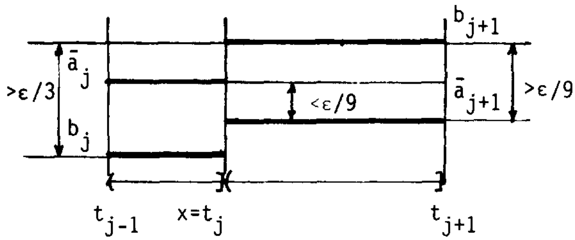

FIGURE 1

Case 2: $x = t_j$ for some $j < n$. Then it follows from (38) that

$$|\bar{f}_{\pi,p}(x) - \bar{f}_{\pi,p}(y)| = |b_j - b_j| = 0$$

for all $y \in y(t_{j-1}, x]$. Thus, let us consider $y \in (x, t_{j+1}]$ and suppose that

$$|\bar{f}_{\pi,p}(y) - \bar{f}_{\pi,p}(x)| = \bar{f}_{\pi,p}(y) - \bar{f}_{\pi,p}(x) = b_{j+1} - b_j > \varepsilon/3.$$

Then we obtain (Fig. 1)

$$\varepsilon/3 < b_{j+1} - b_j = (b_{j+1} - \bar{a}_{j+1}) + (\bar{a}_{j+1} - \bar{a}_j) + (\bar{a}_j - b_j)$$

since $(\bar{a}_{j+1} - \bar{a}_j) < \varepsilon/9$ by (35); then we may assume without loss of generality that

$$b_{j+1} - \bar{a}_{j+1} > \varepsilon/9. \tag{39}$$

In this case let

$$b^*_{j+1} = b_{j+1} - \varepsilon/9. \tag{40}$$

Hence

$$\begin{aligned} b^*_{j+1} - b_j &= (b_{j+1} - b_j) - \varepsilon/9 \\ &> \varepsilon/3 - \varepsilon/9 = 2\varepsilon/9 > 0. \end{aligned}$$

Let $\bar{f}^*_{\pi,p}$ be the monotone increasing step function defined by

$$\begin{aligned} \bar{f}^*_{\pi,p} = b_1 I_{[t_0,t_1]} &+ \sum_{i=2}^{j} b_i I_{(t_{i-1},t_i]} \\ &+ b^*_{j+1} I_{(t_j,t_{j+1}]} + \sum_{i=j+2}^{n} b_i I_{(t_{i-1},t_i]}. \end{aligned}$$

Then

$$\|\bar{f}^*_{\pi,p} - \bar{f}_\pi\|_p^p = \sum_{i=1}^{j} (t_i - t_{i-1})\,|b_i - \bar{a}_i|^p + (t_{j+1} - t_j)\,|b^*_{j+1} - \bar{a}_{j+1}|^p$$
$$+ \sum_{i=j+2}^{n} (t_i - t_{i-1})\,|b_i - \bar{a}_i|^p, \tag{41}$$

while

$$\|\bar{f}_{\pi,p} - \bar{f}_\pi\|_p^p = \sum_{i=1}^{n} (t_i - t_{i-1})\,|b_i - \bar{a}_i|^p; \tag{42}$$

but observe that (39) and (40) imply that

$$\begin{aligned} b^*_{j+1} - \bar{a}_{j+1} &= b_{j+1} - \varepsilon/9 - \bar{a}_{j+1} \\ &= (b_{j+1} - \bar{a}_{j+1}) - \varepsilon/9 > \varepsilon/9 - \varepsilon/9 = 0 \end{aligned}$$

or

$$0 < b^*_{j+1} - \bar{a}_{j+1} < b_{j+1} - \bar{a}_{j+1}$$

or

$$|b^*_{j+1} - \bar{a}_{j+1}|^p < |b_{j+1} - \bar{a}_{j+1}|^p,$$

which implies by comparing (41) and (42) that

$$\|\bar{f}^*_{\pi,p} - \bar{f}_\pi\|_p < \|\bar{f}_{\pi,p} - \bar{f}_\pi\|_p.$$

Contradiction! Therefore our assumption is false and hence we conclude that

$$|\bar{f}_{\pi,p}(y) - \bar{f}_{\pi,p}(x)| < \varepsilon/3$$

for all $y \in (x, t_{j+1}]$. Take $\delta = \min\{(x - t_{j-1}),\ (t_{j+1} - x)\}$ to conclude that (36) becomes

$$|f_p(x) - f_p(y)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$$

for all $y \in (x - \delta,\ x + \delta)$. This completes the proof. ∎

COROLLARY 2. *The function* $f_\infty = \lim_{p\to\infty} f_p$ *is continuous when* f *is continuous.*

Proof. Since f_∞ is the uniform limit of continuous functions, it must be continuous. ∎

EXAMPLE 1. Let f be the real-valued continuous function on $[0, 1]$ defined by

$$\begin{aligned} f(x) &= \sin \tfrac{15}{2}\pi(x - \tfrac{1}{15}), && 0 \leqslant x \leqslant \tfrac{1}{3}, \\ &= 2 \sin \tfrac{15}{2}\pi(x - \tfrac{1}{3}), && \tfrac{1}{3} < x \leqslant \tfrac{3}{5}, \\ &= 15(x - \tfrac{3}{5}), && \tfrac{3}{5} < x \leqslant \tfrac{2}{3}, \\ &= 1 + \sin \tfrac{15}{2}\pi(x(x - \tfrac{2}{3}), && \tfrac{2}{3} < x \leqslant 1. \end{aligned}$$

Then f_∞ is the real-valued nondecreasing continuous function given by

$$\begin{aligned} f_\infty(x) &= \sin \tfrac{15}{2}\pi(x - \tfrac{1}{15}), && 0 \leqslant x \leqslant \tfrac{1}{15}, \\ &= 0, && \tfrac{1}{15} < x \leqslant \tfrac{3}{5}, \\ &= 15(x - \tfrac{3}{5}), && \tfrac{3}{5} < x \leqslant \tfrac{2}{3}, \\ &= 1, && \tfrac{2}{3} < x \leqslant \tfrac{14}{15}, \\ &= 1 + \sin \tfrac{15}{2}\pi(x - \tfrac{2}{3}), && \tfrac{14}{15} < x \leqslant 1. \end{aligned}$$

It is shown in [3, p. 664, Theorem 2] that a nondecreasing function g is a best L_∞-approximation to $f \in Q^*$ by elements of M^* if and only if

$$\underline{g} \leqslant g \leqslant \bar{g},$$

where $\underline{g}$ and $\bar{g}$ are given by

$$\underline{g}(x) = \sup\{(f(z) - \theta) \colon z \in [0, x]\}, \qquad x \in [0, 1],$$

and

$$\bar{g}(x) = \inf\{(f(z) + \theta) \colon z \in [x, 1]\}, \qquad x \in [0, 1],$$

where

$$\theta = d(f, M^*) = \inf\{\|f - h\|_\infty : h \in M^*\}.$$

Thus, if f is the function in Example 1, then it is easily seen that

$$d(f, M^*) = 2$$

and hence it follows that

$$\bar{g}(x) = (f(x) + 2)\, I_{(8/15, 3/5]} + 2I_{(3/5, 13/15]} + (f(x) + 2)\, I_{(13/15, 1]}$$

and

$$\underline{g}(x) = (f(x) - 2)\, I_{[0, 2/15]} - I_{(2/15, 16/45]} + (f(x) - 2)\, I_{(16/45, 2/5]}.$$

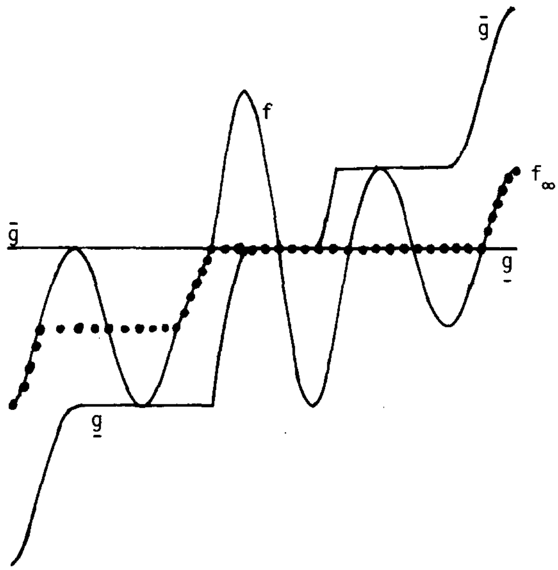

FIGURE 2

Finally, notice that f_∞ is not the average of $\underline{g}$ and $\bar{g}$ on [0, 1], e.g., on $[0, \frac{1}{15}]$ (see Fig. 2)

$$f_\infty \neq \tfrac{1}{2}(\underline{g} + \bar{g}) \qquad \text{everywhere.}$$

EXAMPLE 2. Let f be the real-valued step function defined on [0, 1] by

$$f = 3I_{[0,1/15]} + 5I_{(3/15,4/15]} + 7I_{(8/15,9/15]}.$$

Figure 3 is a sketch of f and the corresponding f_2, f_4, and f_∞. Notice that f_2 is constant while f_4 is increasing and by our earlier results f_p should converge monotonically to

$$f_\infty = \tfrac{3}{2}I_{[0,1/5]} + \tfrac{5}{2}I_{(1/5,8/15]} + \tfrac{7}{2}I_{(8/15,1]}$$

as $p \to \infty$.

Remark 5. If f is given by

$$f = k_1 I_{[0,t_1]} + k_2 I_{(t_2,t_3]} + \cdots + k_n I_{(t_{2(n-1)},t_{2n-1}]}, \tag{43}$$

where

$$2 < k_1 < k_2 < \cdots < k_n,$$

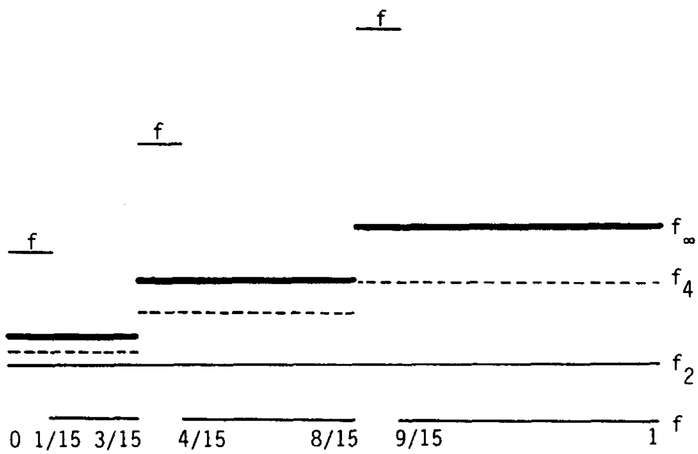

FIGURE 3

and

$$t_1 = \delta = \left(\sum_{j=1}^{n} k_j\right)^{-1},$$

$$t_{2i} = \left(\sum_{j=1}^{i} k_j\right)\delta, \qquad i \geqslant 2,$$

$$t_{2i+1} = t_{2i} + \delta, \qquad i \geqslant 2,$$

then for every p, f_p must have the form

$$f_p = \zeta_1 I_{[0,t_2]} + \zeta_2 I_{(t_2,t_4]} + \cdots + \zeta_n I_{(t_{2(n-1)},1]}, \tag{44}$$

where $0 < \zeta_1 \leqslant \zeta_2 \leqslant \cdots \leqslant \zeta_n$ and ζ_i depends on p for all i.

Suppose we want to compute f_2 which has form (44). Then ζ_1 must be the unique real number minimizing the function

$$g_1(\zeta) = \delta(k_1 - \zeta)^2 + \delta(k_1 - 1)\zeta^2.$$

Differentiating g_1 we obtain

$$g_1'(\zeta_1) = -2\delta(k_1 - \zeta_1) + 2\delta(k_1 - 1)\zeta_1$$
$$= 2\delta k_1(\zeta_1 - 1) = 0.$$

Thus $\zeta_1 = 1$. Similarly ζ_i is the unique real number minimizing the function

$$g_i(\zeta) = \delta(k_i - \zeta)^2 + \delta(k_i - 1)\zeta^2,$$

which implies that $\zeta_i = 1$ for all $i \leqslant n$. Hence $f_2 \equiv 1$ on $[0, 1]$.

Next, let us compute f_p for $p > 2$, where f_p has form (44) and f is given by (43). Then ζ_i will be the unique real number minimizing the function

$$g_i(\zeta) = (k_i - \zeta)^p + (k_i - 1)\zeta^p.$$

Differentiating g_i we obtain

$$g_i'(\zeta_i) = -p(k_i - \zeta_i)^{p-1} + p(k_i - 1)\, \zeta_i^{p-1} = 0.$$

Dividing by $(p\zeta_i^{p-1})$, we obtain

$$(k_i - 1) = (k_i/\zeta_i) - 1)^{p-1}$$

or

$$\zeta_i = k_i/((k_i - 1)^{1/\lambda} + 1), \tag{45}$$

where $\lambda = p - 1$ and $i = 1, 2,..., n$.

Observe that as $\lambda \to \infty$ in (45), $\zeta_i \to k_i/2$, which says that f_p converges to a function f_∞ given by

$$f_\infty = (k_1/2)\, I_{[0,t_2]} + (k_2/2)\, I_{(t_2,t_4]} + \cdots + (k_n/2)\, I_{(t_{n-1},1]}, \tag{46}$$

which is consistent with our definition of f_∞, where f is defined above.

Finally, we show that for a fixed $\lambda > 1$ and a fixed i, the value of $\zeta = \zeta_i$ increases as $k = k_i$ increases. From (45), consider

$$\zeta = \psi(k) = k/((k - 1)^{1/\lambda} + 1).$$

We show that $\psi'(k) > 0$ for all $k > 2$. Thus, letting $\alpha = 1/\lambda$, we have

$$\begin{aligned}\psi'(k) &= \frac{1}{(k-1)^\alpha + 1} - k\left[\frac{(k-1)^{\alpha-1}}{((k-1)^\alpha + 1)^2}\right] \\ &= \frac{1}{(k-1)^\alpha + 1}\left[1 - \frac{\alpha k(k-1)^{\alpha-1}}{(k-1)^\alpha + 1}\right].\end{aligned}$$

To show that $\psi'(k) > 0$, all we need to show is that

$$\frac{\alpha k(k-1)^{\alpha-1}}{(k-1)^\alpha + 1} < 1. \tag{47}$$

But indeed we have

$$\frac{\alpha k(k-1)^{\alpha-1}}{(k-1)^{\alpha}+1}=\frac{(k-1)^{\alpha}\alpha k(k-1)^{-1}}{(k-1)^{\alpha}(1+1/(k-1)^{\alpha})}$$
$$=\frac{\alpha k}{(k-1)(1+1/(k-1)^{\alpha})}$$
$$=\frac{k}{\lambda(k-1+((k-1)/(k-1)^{\alpha}))}.$$

Since $\alpha=1/\lambda<1$, then $(k-1)/(k-1)^{\alpha}>1$, which implies that $(k-1+((k-1)/(k-1)^{\alpha}))>k$, or

$$\frac{k}{\lambda(k-1+((k-1)/(k-1)^{\alpha}))}<\frac{k}{\lambda k}=\frac{1}{\lambda}<1.$$

Hence, (47) is true and $\psi'(k)>0$.

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